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Stochastic second-order perturbation approach to the stress-based finite element method

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Abstract

The paper is devoted to the application of the second-order perturbation second probabilistic moment method to the stress-based finite element method (FEM). The approach is introduced for the linear elastic heterogeneous medium – up to the second-order, variational equations of the complementary energy principle are presented together with an additional stochastic finite element discretization based on Airy and Prandtl stress functions. The numerical examples shown in this paper illustrate the probabilistic stress and strain tensors in the cantilever beam under shear loading and torsioned square beam with randomly defined material and geometrical parameters. The results obtained in the tests can be applied in probabilistic analyses of the boundary value problems having any closed form mathematical solutions as well as in the stress-based stochastic FEM analysis of solids and structures. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The equilibrium problems of solids and structures with random coefficients or under random excitations were studied very extensively by numerous authors in the context of probabilistic static (Elishakoff et al., 1995; Ghanem and Spanos, 1991; Kleiber and Hien, 1992; Vanmarcke and Grigoriu, 1983; Grigoriu, 2000) and dynamic response (Ghanem and Spanos, 1991; Kleiber and Hien, 1992; Liu et al., 1986; Schüeller and Shinozuka, 1987) both in linear and nonlinear range, taking into account the formulation and verification of various reliability criteria (Der Kiureghian and Jyh, 1988; Thoft-Christensen and Baker, 1982). It was done by the use of different Monte-Carlo simulation (MCS) approaches, stochastic spectral techniques, stochastic weighted residuals or, alternatively, stochastic perturbation techniques. Since the computational

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time savings in comparison with simulation methods (especially in the case of large scale engineering systems discrete modeling) and, on the other hand, taking into account the capability of spatial discretization of random fields, the second-order perturbation second probabilistic moment (SOSM) method is used with its numerical implementation called the stochastic finite element method (SFEM). The method has been successfully applied in displacement-based finite element method (FEM) in elastostatic, elastodynamic as well as inelastic problems for both homogeneous and heterogeneous media (Kaminski and Hien, 1999; Kaminski and Kleiber, 2000; Kleiber and Hien, 1992; Liu et al., 1986). On the other hand, it is known from the homogenization theory fundamental for composite materials modeling (Borkowski, 1977; Wieckowski, 1999) that the displacement-based FEM makes it possible to compute the lower bounds for the effective properties of heterogeneous media while the stress-based approach enables to calculate the corresponding upper values – since that both of them should be implemented. Further, considering the fact, that most of reliability criteria are based on the stress tensor probabilistic moments and taking into account very complicated form of this tensor moments in displacement-based FEM (Kleiber and Hien, 1992; Liu et al., 1986; Zienkiewicz and Taylor, 1991), its stress-based (Azene, 1979; Desai, 1979; Rybicki and Schmit, 1970; Watwood and Hartz, 1968; Wieckowski, 1999) stochastic version is now proposed. The mathematical model, computational discretization and solution of the boundary value problems for the engineering structures with random material and geometrical parameters by using specially utilized stress-based FEM are discussed below.

Due to the traditional general perturbation (Nayfeh, 1973) and the SOSM methodology (Kleiber and Hien, 1992; Liu et al., 1986), up to the second-order variational equations are introduced corresponding to the zeroth-, first- and second-order stress solution are written out and, starting from these equations, the expected values and cross-covariances of all the state functions are derived; all of these relations are obtained starting from the classical definitions of the first two probabilistic moments fundamental for the probability theory (Feller, 1967; Vanmarcke, 1983). Next, the matrix description for the second-order stress-based version of the stochastic FEM is introduced together with detailed equations for the case, where some of the elastic and geometric characteristics are input random variables of the boundary value problem. All the considerations are illustrated with two examples – the two-dimensional (2D) equilibrium problem of the homogeneous cantilever under shear force with some parameters randomized and solved by the global Airy stress functions approach and the computer patch test of the bar with rectangular cross-section in torsion by using the stochastic constant triangle finite elements and Prandtl function. The results of the second example are compared against the MCS that makes it possible to verify the general restrictions on the second-order perturbation technique.

2. Governing equations

2.1. Deterministic problem

Let us consider the set $\Omega \subset \Re^3$ bounded by the regular and sufficiently smooth boundary $\partial\Omega$ and consider a heterogeneous linear elastic medium in Ω built up with n homogeneous and coherent components. Thus, the elasticity tensor is defined as follows:

$$C_{ijkl}(\mathbf{x}) = \chi_a(\mathbf{x})C_{ijkl}^{(a)} \quad \text{for } i,j,k,l = 1,2, \ \ a = 1,\ldots,n, \tag{1}$$

where a is a number of homogeneous components of Ω,

$$C_{ijkl}^{(a)} = \delta_{ij}\delta_{kl}\lambda_a + \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\mu_a \tag{2}$$

with λ, μ being the Lame constants and

$$\chi_a(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \Omega_a \\ 0, & \mathbf{x} \in \Omega/\Omega_a \end{cases} \tag{3}$$

being the characteristic function. By the analogous way, the compliance tensor is introduced as a function of engineering constants, i.e. Young's moduli and Poisson's coefficients for Ω constituents as follows

$$c_{ijkl}^{(a)} = \frac{1}{E_a}\left((1+\nu_a)\delta_{ik}\delta_{jl} - \nu_a(1+\kappa\nu_a)\delta_{ij}\delta_{kl}\right), \tag{4}$$

where

$$c_{ijkl}(\mathbf{x}) = \chi_a(\mathbf{x})c_{ijkl}^{(a)} \quad \text{for } i,j,k,l = 1,2, \;\; a = 1,\ldots,n, \tag{5}$$

and with $\kappa = 0{,}1$ for the plane stress or the plane strain problem. Next, the boundary value problem is defined on Ω using the following equations:

$$\sigma_{ij,j} = 0, \quad \mathbf{x} \in \Omega, \tag{6}$$

$$\varepsilon_{ij} = c_{ijkl}\sigma_{kl}, \quad \mathbf{x} \in \Omega, \tag{7}$$

$$\varepsilon_{ij} = \tfrac{1}{2}\left(u_{i,j} + u_{j,i}\right), \quad \mathbf{x} \in \Omega, \tag{8}$$

$$u_i = \hat{u}_i, \quad \mathbf{x} \in \partial\Omega_u, \tag{9}$$

$$\sigma_{ij}n_j = \hat{t}_i, \quad \mathbf{x} \in \partial\Omega_\sigma, \tag{10}$$

where

$$\partial\Omega_\sigma \cup \partial\Omega_u = \partial\Omega, \tag{11}$$

$$\partial\Omega_\sigma \cap \partial\Omega_u = \emptyset. \tag{12}$$

The variational formulation for the boundary value problem can be proposed (Borkowski, 1977; Tabarrok, 1984; Wieckowski, 1999) by using the statically admissible stresses space S_0 such that

$$S_0 = \left\{\mathbf{s} \in \left[L^2(\Omega)\right]^9 : \sigma_{ij} = \sigma_{ji}; \sigma_{ij,j} = 0 : \mathbf{x} \in \Omega; \sigma_{ij}n_j = 0 : \mathbf{x} \in \partial\Omega_\sigma\right\}. \tag{13}$$

Multiplying Eq. (8) by the stress variation $\delta\sigma_{ij}$ and integrating the result over the region Ω we arrive at

$$\int_\Omega \left(\varepsilon_{ij} - \frac{1}{2}(u_{i,j} + u_{j,i})\right)\delta\sigma_{ij}\,\mathrm{d}\Omega = 0. \tag{14}$$

Applying the Green formula to displacement field components u_i, the complementary energy principle is obtained for any $\delta\sigma_{ij} \in S_0$ in the form of

$$\int_\Omega \varepsilon_{ij}\delta\sigma_{ij}\,\mathrm{d}\Omega - \int_{\partial\Omega_u} \hat{u}_i\delta\sigma_{ij}n_j\,\mathrm{d}(\partial\Omega) = 0 \tag{15}$$

or, alternatively, by using the following functional:

$$\Sigma(\sigma) = \frac{1}{2}\int_\Omega c_{ijkl}\sigma_{ij}\sigma_{kl}\,\mathrm{d}\Omega - \int_{\partial\Omega_u} \hat{u}_i\sigma_{ij}n_j\,\mathrm{d}(\partial\Omega), \tag{16}$$

which, after minimization, leads to the real stress field being a solution for equilibrium problem (6)–(12). The equations posed above enables one to solve the boundary problem with random coefficients using the SOSM method, which is shown in the next paragraph. It should be underlined that the method has quite a general character and makes it possible to randomize any differential or algebraic equations with respect to up to the second-order perturbations of the problem parameters as well as up to the second-order probabilistic moments of all the state variables.

Furthermore, quite analogous formulation may be proposed in the analysis of torsion of a linear, isotropic and homogeneous medium formulated in terms of a warping function φ (Desai, 1979):

$$\frac{1}{G}\left(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\right)=Q(x,y), \tag{17}$$

where $Q(x,y)$ represents an external load. The stress tensor components can be introduced as

$$\sigma_{13}=\frac{\partial\varphi}{\partial x_2}, \qquad \sigma_{23}=-\frac{\partial\varphi}{\partial x_1}, \tag{18}$$

while the constitutive relation can be rewritten as

$$\frac{\partial\varphi}{\partial x_2}=G\theta\left(\frac{\partial\psi}{\partial x_1}-x_2\right), \tag{19}$$

$$\frac{\partial\varphi}{\partial x_1}=G\theta\left(\frac{\partial\psi}{\partial x_2}+x_1\right), \tag{20}$$

in the case of an externally applied twisting moment $Q(x,y)=-2\theta$ with Ψ being a warping function. Finally, the complementary energy necessary to finite element discretization can be expressed as

$$\Sigma(\varphi)=\int_\Omega\frac{1}{2G}\left[\left(\frac{\partial\varphi}{\partial x_1}\right)^2+\left(\frac{\partial\varphi}{\partial x_2}\right)^2\right]\mathrm{d}\Omega-\int_{\partial\Omega_u}Q\varphi\,\mathrm{d}\Omega. \tag{21}$$

As it is known, the torsion problem is so-called field problem and is equivalent to the heat conduction, seepage and another related physical phenomena described by the same Laplace partial differential equation; solution of this particular problem makes it possible to describe, at the same time, all equivalent problems in the context of the field analogies.

2.2. Second-order perturbation second probabilistic moment approach

Let us denote random variable of the problem as vector $\{b^r(\mathbf{x};\omega)\}$ and its probability density as $g(b^r)$ and $g(b^r,b^s)$ respectively ($r,s=1,2,\ldots,R$ indexing various random variables). The expected values of this variable is defined as follows (Feller, 1967; Vanmarcke, 1983):

$$E[b^r]=\int_{-\infty}^{+\infty}b^r g(b^r)\,\mathrm{d}b^r, \tag{22}$$

while the covariance

$$S_b^{rs}=\mathrm{cov}(b^r,b^s)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(b^r-E[b^r])(b^s-E[b^s])g(b^r,b^s)\,\mathrm{d}b^r\mathrm{d}b^s. \tag{23}$$

If the discrete representation of random vector $\boldsymbol{b}(\mathbf{x};\omega)$ is used (in terms of experimental data), the statistical estimators (Bendat and Piersol, 1971) may be applied to approximate any order probabilistic moments of the vector.

Next, all material and physical parameters of Ω as well as the state functions (random fields resulting from equilibrium problem solution) are extended in the variational formulation by the use of the following stochastic Taylor series expansion:

$$F(\mathbf{x};\omega) = F^0(\mathbf{x};\omega) + \sum_{n=1}^{N}\left\{\frac{\theta^n}{n!}F^{(n)}(\mathbf{x};\omega)\prod_{i=1}^{n}\Delta b^{r_i}(\omega)\right\}, \tag{24}$$

where θ is given small perturbation, $\theta\Delta b^{r_1}$ denotes the first order variation of Δb^{r_1} about its expected value $E[b^{r_1}]$ and $F^{(n)}(\mathbf{x};\omega)$ represents the nth order partial derivatives with respect to the random variables evaluated at their expected values. Considering great complexity of general nth order perturbation equations and corresponding computational implementation, the second-order approach under some restrictions of input random variables is usually proposed. Hence, the random function $F(\mathbf{x};\omega)$ analyzed is extended as follows:

$$F(\mathbf{x};\omega) = F^0(\mathbf{x};\omega) + \theta F^{,r}(\mathbf{x};\omega)\Delta b^r + \tfrac{1}{2}\theta^2 F^{,rs}(\mathbf{x};\omega)\Delta b^r\Delta b^s. \tag{25}$$

To obtain the SOSM model for the stress-based FEM, Eq. (15) is rewritten as

$$\int_\Omega c_{ijkl}\sigma_{kl}\delta\sigma_{ij}\,\mathrm{d}\Omega = \int_{\partial\Omega_u}\hat{u}_i\delta\sigma_{ij}n_j\,\mathrm{d}(\partial\Omega). \tag{26}$$

Introducing the second-order perturbation terms and equating the components of the same order, the zeroth-, first- and second-order variational statements are obtained as

- zeroth-order (ε^0 terms, one equation):

$$\int_\Omega c^0_{ijkl}\sigma^0_{kl}\delta\sigma_{ij}\,\mathrm{d}\Omega = \int_{\partial\Omega_u}\hat{u}^0_i\delta\sigma_{ij}n_j\,\mathrm{d}(\partial\Omega), \tag{27}$$

- first-order (ε^1 terms, R equations):

$$\int_\Omega c^0_{ijkl}\sigma^{,r}_{kl}\delta\sigma_{ij}\,\mathrm{d}\Omega = \int_{\partial\Omega_u}\hat{u}^{,r}_i\delta\sigma_{ij}n_j\,\mathrm{d}(\partial\Omega) - \int_\Omega c^{,r}_{ijkl}\sigma^0_{kl}\delta\sigma_{ij}\,\mathrm{d}\Omega, \tag{28}$$

- second-order (ε^2 terms, one equation):

$$\int_\Omega c^0_{ijkl}\sigma^{,rs}_{kl}S^{rs}_b\delta\sigma_{ij}\,\mathrm{d}\Omega = \int_{\partial\Omega_u}\hat{u}^{,rs}_i S^{rs}_b\delta\sigma_{ij}n_j\,\mathrm{d}(\partial\Omega) - \int_\Omega\left(2c^{,r}_{ijkl}\sigma^{,s}_{kl} + c^{,rs}_{ijkl}\sigma^0_{kl}\right)S^{rs}_b\delta\sigma_{ij}\,\mathrm{d}\Omega. \tag{29}$$

It should be noticed that the second-order equation is obtained above by multiplying the R-variate probability density function $p_R(b_1,b_2,\ldots,b_R)$ by the ε^2-terms and integrating over random vector $\mathbf{b}(x_k)$ domain. There holds, for instance

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\left[\int_{\partial\Omega_u}\varepsilon^2\hat{u}^{,rs}_i\left[\mathbf{b}^0(x_k);x_k\right]\Delta b_r(x_k)\Delta b_s(x_k)\delta\sigma_{ij}n_j\,\mathrm{d}(\partial\Omega)\right]p_R(\mathbf{b}(x_k))\,\mathrm{d}\mathbf{b}\\
&= \varepsilon^2\int_{\partial\Omega_u}\hat{u}^{,rs}_i\left[\mathbf{b}^0(x_k);x_k\right]\left[\int_{-\infty}^{+\infty}\Delta b_r(x_k)\Delta b_s(x_k)p_R(\mathbf{b}(x_k))\,\mathrm{d}\mathbf{b}\right]\delta\sigma_{ij}n_j\,\mathrm{d}(\partial\Omega)\\
&= \varepsilon^2\int_{\partial\Omega_u}\hat{u}^{,rs}_i\left[\mathbf{b}^0(x_k);x_k\right]S^{rs}_b\delta\sigma_{ij}n_j\,\mathrm{d}(\partial\Omega).
\end{aligned} \tag{30}$$

Further, it can be observed that the occurrence of the double sums $(\cdot)^{,rs}S^{rs}_b$ and $(\cdot)^{,r}(\cdot)^{,s}S^{rs}_b$ in the above formulation makes it possible to obtain the single equation of the second-order (29), while without this closure $R(R+1)/2$ analogous equations must be solved since the symmetry of stiffness matrix. Let us note that for nonsymmetric problems (Boundary Element Method formulation), R^2 second-order equations must be solved.

To determine the probabilistic solution for the equilibrium problem considered, Eq. (27) is solved for σ^0_{kl}, next Eq. (28) – for the first-order terms of σ_{kl} and, finally, Eq. (29) for $\sigma^{(2)}_{kl}$. The probabilistic two moment characterization of all the state functions begins with the expected value of the stress tensor components. Using the definition and introducing second-order expansion, gives

$$\begin{aligned} E[\sigma_{kl}[\mathbf{b}(x_k);x_k]] &= \int_{-\infty}^{+\infty} \sigma_{kl}[\mathbf{b}(x_k);x_k] p_R(\mathbf{b}(x_k))\,\mathrm{d}\mathbf{b} \\ &= \int_{-\infty}^{+\infty} \Big\{ \sigma^0_{kl}[\mathbf{b}(x_k);x_k] + \sigma^{,r}_{kl}[\mathbf{b}(x_k);x_k]\Delta b_r(x_k) \\ &\qquad + \frac{1}{2}\sigma^{,rs}_{kl}[\mathbf{b}(x_k);x_k]\Delta b_r(x_k)\Delta b_s(x_k) \Big\} p_R(\mathbf{b}(x_k))\,\mathrm{d}\mathbf{b} \end{aligned} \tag{31}$$

where $\varepsilon = 1$ is applied, and, further this becomes

$$\begin{aligned} &\sigma^0_{kl}(x_k)\int_{-\infty}^{+\infty} p_R(\mathbf{b}(x_k))\,\mathrm{d}\mathbf{b} + \sigma^{,r}_{kl}(x_k)\int_{-\infty}^{+\infty} \Delta b_r(x_k) p_R(\mathbf{b}(x_k))\,\mathrm{d}\mathbf{b} \\ &\quad + \frac{1}{2}\sigma^{,rs}_{kl}(x_k)\int_{-\infty}^{+\infty} \Delta b_r(x_k)\Delta b_s(x_k) p_R(\mathbf{b}(x_k))\,\mathrm{d}\mathbf{b}. \end{aligned} \tag{32}$$

This result leads to the following conclusion:

$$E[\sigma_{kl}[\mathbf{b}(x_k);x_k]] = \sigma^0_{kl}[\mathbf{b}(x_k);x_k] + \tfrac{1}{2}\sigma^{,rs}_{kl}[\mathbf{b}(x_k);x_k]S^{rs}_b. \tag{33}$$

Next, the first-order cross-covariances for the stress tensor components are derived as follows:

$$\begin{aligned} \mathrm{cov}\Big(\sigma_{\alpha\beta}\Big[\mathbf{b}\Big(x_k^{(1)}\Big);x_k^{(1)}\Big];\sigma_{\gamma\delta}\Big[\mathbf{b}\Big(x_k^{(2)}\Big);x_k^{(2)}\Big]\Big) &= S^{\alpha\beta\gamma\delta}_\sigma\Big(x_k^{(1)};x_k^{(2)}\Big) \\ &= \int_{-\infty}^{+\infty}\Big\{\sigma_{\alpha\beta}\Big[\mathbf{b}\Big(x_k^{(1)}\Big);x_k^{(1)}\Big] - E\Big[\sigma_{\alpha\beta}\Big[\mathbf{b}\Big(x_k^{(1)}\Big);x_k^{(1)}\Big]\Big]\Big\} \\ &\quad \times \Big\{\sigma_{\gamma\delta}\Big[\mathbf{b}\Big(x_k^{(2)}\Big);x_k^{(2)}\Big] - E\Big[\sigma_{\gamma\delta}\Big[\mathbf{b}\Big(x_k^{(2)}\Big);x_k^{(2)}\Big]\Big]\Big\} p_R(\mathbf{b}(x_k))\,\mathrm{d}\mathbf{b}. \end{aligned} \tag{34}$$

Therefore,

$$S^{\alpha\beta\gamma\delta}_\sigma\Big(x_k^{(1)};x_k^{(2)}\Big) = \sigma^{,r}_{\alpha\beta}\Big(x_k^{(1)}\Big)\sigma^{,s}_{\gamma\delta}\Big(x_k^{(2)}\Big)S^{rs}_b. \tag{35}$$

To determine the first two probabilistic moments for the strains, the second-order perturbations are inserted in constitutive relation (7) as

$$\begin{aligned} \varepsilon_{ij} &= c_{ijkl}\sigma_{kl} \\ &= (c^0_{ijkl}[b(x_k);x_k] + c^{,r}_{ijkl}[b(x_k);x_k]\Delta b_r(x_k) + \tfrac{1}{2}c^{,rs}_{ijkl}[b(x_k);x_k]\Delta b_r(x_k)\Delta b_s(x_k)) \\ &\quad \times (\sigma^0_{kl}[b(x_k);x_k] + \sigma^{,u}_{kl}[b(x_k);x_k]\Delta b_r(x_k) + \tfrac{1}{2}\sigma^{,uv}_{kl}[b(x_k);x_k]\Delta b_r(x_k)\Delta b_s(x_k)) \end{aligned} \tag{36}$$

for $r,s,u,v = 1,\ldots,R$. Since that, the second-order expected value for the strains can be derived as

$$E\big[\varepsilon_{ij}[\mathbf{b}(x_k);x_k]\big] = c^0_{ijkl}(x_k)\sigma^0_{kl}(x_k) + \tfrac{1}{2}\Big[c^{,rs}_{ijkl}(x_k)\sigma^0_{kl}(x_k) + 2c^{,r}_{ijkl}(x_k)\sigma^{,s}_{kl}(x_k) + c^0_{ijkl}(x_k)\sigma^{,rs}_{kl}(x_k)\Big]S^{rs}_b, \tag{37}$$

while the first-order accurate cross-covariance has the form

$$\begin{aligned}\mathrm{cov}\left(\varepsilon_{ij}\left[\mathbf{b}\left(x_k^{(1)}\right);x_k^{(1)}\right];\varepsilon_{kl}\left[\mathbf{b}\left(x_k^{(2)}\right);x_k^{(2)}\right]\right) &= S_\varepsilon^{ijkl}\left(x_k^{(1)};x_k^{(2)}\right)\\ &= \left[c_{ijmn}^{,r}\left(x_k^{(1)}\right)c_{\overline{ijmn}}^{,s}\left(x_k^{(2)}\right)\sigma_{mn}^{0}\left(x_k^{(1)}\right)\sigma_{\overline{mn}}^{0}\left(x_k^{(2)}\right)\right.\\ &\quad + c_{ijmn}^{,r}\left(x_k^{(1)}\right)c_{\overline{ijmn}}^{0}\left(x_k^{(2)}\right)\sigma_{mn}^{0}\left(x_k^{(1)}\right)\sigma_{\overline{mn}}^{,s}\left(x_k^{(2)}\right)\\ &\quad + c_{ijmn}^{0}\left(x_k^{(1)}\right)c_{\overline{ijmn}}^{,r}\left(x_k^{(2)}\right)\sigma_{mn}^{,s}\left(x_k^{(1)}\right)\sigma_{\overline{mn}}^{0}\left(x_k^{(2)}\right)\\ &\quad \left.+ c_{ijmn}^{0}\left(x_k^{(1)}\right)c_{\overline{ijmn}}^{0}\left(x_k^{(2)}\right)\sigma_{mn}^{,r}\left(x_k^{(1)}\right)\sigma_{\overline{mn}}^{,s}\left(x_k^{(2)}\right)\right]S_b^{rs}.\end{aligned} \tag{38}$$

By the quite analogous way, the expected values and the cross-covariances of the displacement field components can be derived, which completes the second-order second moment characterization of the stress-based solution for the linear elastostatics equilibrium problem.

3. Stress-based finite element method

3.1. Deterministic approach

The following approximation of the stress tensor components in terms of Airy functions is applied in the case of 2D problems to discretize variational statement (15):

$$\mathbf{s} = \begin{bmatrix} \frac{\partial^2}{\partial y^2} \\ \frac{\partial^2}{\partial x^2} \\ -\frac{\partial^2}{\partial x \partial y} \end{bmatrix} \mathbf{F} = \partial^2 \mathbf{F}, \tag{39}$$

where function $F(x,y)$ may be represented as follows:

$$F(x,y) = \sum_{p,q \leqslant \alpha} d_{pq} x^p y^q \tag{40}$$

with the value of parameter α depending on the type of the stress-based finite element being used. Next, the interpolation function $\mathbf{N}$ is introduced with the degrees of freedom vector $\mathbf{a}$ what makes it possible to apply the following representation:

$$\mathbf{s} = \partial^2 \mathbf{N} \mathbf{a} = \begin{bmatrix} \frac{\partial^2 N_1}{\partial y^2} & \frac{\partial^2 N_2}{\partial y^2} & \cdots \\ \frac{\partial^2 N_1}{\partial x^2} & \frac{\partial^2 N_2}{\partial x^2} & \cdots \\ -\frac{\partial^2 N_1}{\partial x \partial y} & -\frac{\partial^2 N_2}{\partial x \partial y} & \cdots \end{bmatrix} \mathbf{a} = \mathbf{H}\mathbf{a}. \tag{41}$$

Defining the prescribed displacements vector $\hat{u}_i$ for $\mathbf{x} \in \partial\Omega_u$ and the matrix

$$\mathbf{n} = \begin{bmatrix} n_x & 0 & n_x \\ 0 & n_y & n_y \end{bmatrix}, \tag{42}$$

with n_x and n_y denoting the components of the unity vector being normal to $\partial\Omega$ and directed externally to Ω, Eq. (15) may be rewritten as

$$\delta \mathbf{a}^{\mathrm{T}} \int_\Omega \mathbf{H}^{\mathrm{T}} \mathbf{a} \, \mathrm{d}\Omega - \delta \mathbf{a}^{\mathrm{T}} \int_{\partial\Omega_u} \mathbf{H}^{\mathrm{T}} \mathbf{n}^{\mathrm{T}} \hat{\mathbf{u}} \, \mathrm{d}(\partial\Omega) = 0, \tag{43}$$

and hence

$$\int_{\Omega} \mathbf{H}^{\mathrm{T}} \mathbf{a} \,\mathrm{d}\Omega - \mathbf{F} = \mathbf{0}, \tag{44}$$

where

$$\mathbf{F} = \int_{\partial\Omega_u} \mathbf{H}^{\mathrm{T}} \mathbf{n}^{\mathrm{T}} \hat{\mathbf{u}} \,\mathrm{d}(\partial\Omega). \tag{45}$$

Finally, analogously to displacement formulation of the FEM, it can be written that

$$\mathbf{Ka} = \mathbf{F}, \tag{46}$$

where $\mathbf{K}$ is a system compliance matrix defined as

$$\mathbf{K} = \int_{\Omega} \mathbf{H}^{\mathrm{T}} \mathbf{cH} \,\mathrm{d}\Omega. \tag{47}$$

It should be mentioned that in general case, the stress tensor components may be rewritten as

$$\sigma_{ij} = \varepsilon_{ipr} \varepsilon_{jqs} \Phi_{rs,pq} \tag{48}$$

with Φ_{rs} denoting the Maxwell–Morera function components; further, computational aspects of the stress-based FEM can be found in Wieckowski, (1999).

Taking into account the torsion problem discussed in Section 2.1 (cf. Eqs. (17)–(21)) and the FEM discretization by the use of triangular finite element (constant stress triangle finite element – CST (Desai, 1979)), the following description for warping function φ is obtained:

$$\varphi = N_1 \varphi_1 + N_2 \varphi_2 + N_3 \varphi_3 = \mathbf{Na}. \tag{49}$$

Therefore, the complementary energy is obtained as

$$\Sigma(\varphi) = \frac{1}{2} \int \int_{\Omega} \mathbf{a}^{\mathrm{T}} \mathbf{H}^{\mathrm{T}} \mathbf{dHa} \,\mathrm{d}\Omega - \int \int_{\Omega} 2\theta \mathbf{Ha} \,\mathrm{d}\Omega. \tag{50}$$

Minimization of the functional $\Sigma(\varphi)$ results in

$$\int \int_{\Omega} \mathbf{H}^{\mathrm{T}} \mathbf{dH} \,\mathrm{d}\Omega \mathbf{a} = \int \int_{\Omega} 2\theta \mathbf{N}^{\mathrm{T}} \,\mathrm{d}\Omega, \tag{51}$$

what gives as a result Eq. (46) with

$$\mathbf{K} = \frac{1}{G} \int \int_{\Omega} \mathbf{B}^{\mathrm{T}} \mathbf{B} \,\mathrm{d}\Omega \tag{52}$$

for a homogeneous region characterized by Kirchhoff modulus G and external load vector given in terms of twisting angle 2θ as

$$\mathbf{F} = 2\theta \int \int_{\Omega} \mathbf{N}^{\mathrm{T}} \,\mathrm{d}\Omega. \tag{53}$$

In the case of heterogeneous medium, the integration in Eq. (52) should be carried out over all components for $a = 1, \ldots, n$.

3.2. Stochastic stress-based finite elements

To introduce the matrix equations for the second-order and second moment stochastic analysis, let us consider a space discretization of Ω by a typical finite element mesh. First, the input vector of random variables $b_r(\mathbf{x})$ is discretized in terms of some points values using the following spatial representation:

$$b_r(\mathbf{x}) = \varphi_{r\alpha}(\mathbf{x}) b_\alpha, \quad r = 1, \ldots, R, \;\; \alpha = 1, \ldots, N, \tag{54}$$

where α is the shape function for αth node, N is the total number of nodal points in the mesh, while $b_{r\alpha}$ is the matrix of random parameters nodal values. Then, the expected values and cross-covariances are interpolated as

$$E[b_r(\mathbf{x})] = b_r^0(\mathbf{x}) = \varphi_{r\alpha}(\mathbf{x}) b_\alpha^0, \tag{55}$$

$$\mathrm{cov}(b_r(\mathbf{x}); b_s(\mathbf{x})) = S_b^{rs} = \varphi_{r\rho}(\mathbf{x})\varphi_{s\sigma}(\mathbf{x}) S_b^{\rho\sigma}, \quad \rho, \sigma = 1, \ldots, N, \tag{56}$$

where

$$\Delta b_r(\mathbf{x}) = \varphi_{r\alpha}(\mathbf{x}) \Delta b_\alpha, \tag{57}$$

$$\Delta b_\alpha = b_\alpha - b_\alpha^0, \tag{58}$$

b_ρ^0 and $S_b^{\rho\sigma}$ are the random value vector and the covariance matrix of this vector. It should be emphasized that the node-type spatial discretization of random fields can be introduced as equivalent to the nodal points of an original mesh. Alternatively, the same discretization may be carried out by using the additional averaging method (random variable is defined as the spatial average of the random field over the single finite element domain), the midpoint method (random variable is defined as the value of the random field at the centroid of the element) and, at last, the series expansion method can be applied where the random field is modeled as the series of shape functions with random coefficients and any field discretization (Kleiber and Hien, 1992).

Next, all material properties and the state variables are expanded using the same shape functions – that is illustrated on the example of the compliance tensor

$$c_{ijkl}[b_r(\mathbf{x}); \mathbf{x}] = c_{ijkl}\left[\varphi_{r\rho}(\mathbf{x}) b_\rho; \mathbf{x}\right] = \varphi_\alpha(\mathbf{x}) c_{ijkl\alpha}(b_\rho), \quad \rho = 1, \ldots, R. \tag{59}$$

Moreover, the Taylor series expansion for the nodal random variables is employed in the form of

$$c_{ijkl\alpha}(b_\rho) = c_{ijkl\alpha}^0\left(b_\rho^0\right) + \varepsilon c_{ijkl\alpha}^{;\rho}\left(b_\rho^0\right)\Delta b_\rho + \tfrac{1}{2}\varepsilon^2 c_{ijkl\alpha}^{;\rho\sigma}\left(b_\rho^0\right)\Delta b_\rho \Delta b_\sigma \tag{60}$$

and, since both of these equations, it is obtained that

$$c_{ijkl\alpha}[b_r(\mathbf{x}); \mathbf{x}] = \varphi_\alpha(\mathbf{x})\left(c_{ijkl\alpha}^0\left(b_\rho^0\right) + \varepsilon c_{ijkl\alpha}^{;\rho}\left(b_\rho^0\right)\Delta b_\rho + \tfrac{1}{2}\varepsilon^2 c_{ijkl\alpha}^{;\rho\sigma}\left(b_\rho^0\right)\Delta b_\rho \Delta b_\sigma\right). \tag{61}$$

Hence, up to the second-order perturbations of the compliance tensor are equal to

$$c_{ijkl\alpha}^0[b_r(\mathbf{x}); \mathbf{x}] = \varphi_\alpha(\mathbf{x}) c_{ijkl\alpha}^0\left(b_\rho^0\right), \tag{62}$$

$$c_{ijkl\alpha}^{;r}[b_r(\mathbf{x}); \mathbf{x}]\varphi_{r\rho}(\mathbf{x}) = \varphi_\alpha(\mathbf{x}) c_{ijkl\alpha}^{;\rho}\left(b_\rho^0\right), \tag{63}$$

$$c_{ijkl\alpha}^{;rs}[b_r(\mathbf{x}); \mathbf{x}]\varphi_{r\rho}(\mathbf{x})\varphi_{s\sigma}(\mathbf{x}) = \varphi_\alpha(\mathbf{x}) c_{ijkl\alpha}^{;\rho}\left(b_\rho^0\right). \tag{64}$$

It should be emphasized that this type of discretization can be applied for homogeneous media, while midpoint method is relevant for heterogeneous solids. Applying the above finite element approximations into zeroth-, first- and second-order variational statements, the following hierarchical equilibrium equations are obtained:

- zeroth-order (ε^0 terms, one system of N linear simultaneous algebraic equations for $a_\alpha^0(b_\rho^0), \rho = 1, \ldots, R, \alpha = 1, \ldots, N$):

$$K^0_{\alpha\beta}\left(b^0_\rho\right)a^0_\beta\left(b^0_\rho\right) = F^0_\alpha\left(b^0_\rho\right), \tag{65}$$

- first-order (ε^1 terms, R systems of N linear simultaneous algebraic equations for $a^{;\rho}_\alpha(b^0_\rho), \rho = 1,\ldots,R;$ $\alpha = 1,\ldots,N$):

$$K^0_{\alpha\beta}\left(b^0_\rho\right)a^{;\rho}_\beta\left(b^0_\rho\right) = F^{;\rho}_\alpha\left(b^0_\rho\right) - K^{;\rho}_{\alpha\beta}\left(b^0_\rho\right)a^0_\beta\left(b^0_\rho\right), \tag{66}$$

- second-order (ε^2 terms, one system of N linear simultaneous algebraic equations for $a^{(2)}_\alpha(b^0_\rho), \rho = 1,\ldots,R; \alpha = 1,\ldots,N$):

$$K^0_{\alpha\beta}\left(b^0_\rho\right)a^{(2)}_\beta\left(b^0_\rho\right) = \left[F^{;\rho\sigma}_\alpha\left(b^0_\rho\right) - 2K^{;\rho}_{\alpha\beta}\left(b^0_\rho\right)a^{;\sigma}_\beta\left(b^0_\rho\right) - K^{;\rho\sigma}_{\alpha\beta}\left(b^0_\rho\right)a^0_\beta\left(b^0_\rho\right)\right]S^{\rho\sigma}_b \tag{67}$$

with

$$a^{(2)}_\beta\left(b^0_\rho\right) = a^{;\rho\sigma}_\beta\left(b^0_\rho\right)S^{\rho\sigma}_b. \tag{68}$$

Solving these equations for zeroth-, first- and second-order stress tensor field and applying the extension

$$a_\alpha\left(b_\rho\right) = a^0_\alpha\left(b^0_\rho\right) + \varepsilon a^{;\rho}_\alpha\left(b^0_\rho\right)\Delta b_\rho + \tfrac{1}{2}\varepsilon^2 a^{;\rho\sigma}_\alpha\left(b^0_\rho\right)\Delta b_\rho \Delta b_\sigma, \tag{69}$$

the expected values and cross-covariances can be calculated as

$$E[a_\alpha] = a^0_\alpha + \tfrac{1}{2}a^{;\rho\sigma}_\alpha S^{\rho\sigma}_b, \tag{70}$$

$$S^{\alpha\beta}_a = a^{;\rho}_\alpha a^{;\sigma}_\beta S^{\rho\sigma}_b. \tag{71}$$

Using analogous methodology, the expected values of strain tensor components can be derived as follows:

$$E\left[\varepsilon_{ij}(\mathbf{x})\right] = c^0_{ijkl}(\mathbf{x})H_{kl\alpha}(\mathbf{x})a^0_\alpha + \tfrac{1}{2}\left[c^{;rs}_{ijkl}(\mathbf{x})H_{kl\alpha}(\mathbf{x})a^0_\alpha + 2c^{;r}_{ijkl}(\mathbf{x})H_{kl\alpha}(\mathbf{x})a^{;s}_\alpha + c^0_{ijkl}(\mathbf{x})H_{kl\alpha}(\mathbf{x})a^{;rs}_\alpha\right]S^{rs}_b. \tag{72}$$

At the same time, the first-order strain tensor components cross-covariance has the following form:

$$\begin{aligned}\mathrm{cov}(\varepsilon_{ij}(\mathbf{x});\varepsilon_{kl}(\mathbf{x})) = \Big[&c^{;r}_{ijmn}(\mathbf{x})c^{;s}_{ij\overline{mn}}(\mathbf{x})H_{mn\alpha}(\mathbf{x})a^0_\alpha H_{\overline{mn}\beta}(\mathbf{x})a^0_\beta + c^{;r}_{ijmn}(\mathbf{x})c^0_{ij\overline{mn}}(\mathbf{x})H_{mn\alpha}(\mathbf{x})a^0_\alpha H_{\overline{mn}\beta}(\mathbf{x})a^{;s}_\beta \\ &+ c^0_{ijmn}(\mathbf{x})c^{;r}_{ij\overline{mn}}(\mathbf{x})H_{mn\alpha}(\mathbf{x})a^{;s}_\alpha H_{\overline{mn}\beta}(\mathbf{x})a^0_\beta + c^0_{ijmn}(\mathbf{x})c^0_{ij\overline{mn}}(\mathbf{x})H_{mn\alpha}(\mathbf{x})a^{;r}_\alpha H_{\overline{mn}\beta}(\mathbf{x})a^{;s}_\beta\Big]S^{rs}_b.\end{aligned} \tag{73}$$

Finally, let us consider for illustration, the case where Poisson's coefficient is introduced as the input random variable of the problem. Then, the first- and second-order derivatives of the compliance tensor with respect to this variable can be calculated using Eq. (5) as

$$\frac{\partial c^{(a)}_{ijkl}}{\partial v_a} = \frac{1}{E_a}\left(\delta_{ik}\delta_{jl} - (1 + 2\kappa v_a)\delta_{ij}\delta_{kl}\right), \tag{74}$$

$$\frac{\partial^2 c^{(a)}_{ijkl}}{\partial v^2_a} = \frac{2\kappa}{E_a}\delta_{ij}\delta_{kl}. \tag{75}$$

For the Young moduli of the component materials of Ω treated as random variables, there holds

$$\frac{\partial c^{(a)}_{ijkl}}{\partial E_a} = -\frac{1}{E^2_a}\left((1 + v_a)\delta_{ik}\delta_{jl} - v_a(1 + \kappa v_a)\delta_{ij}\delta_{kl}\right) \tag{76}$$

and, for the second-order partial derivatives with respect to this parameter

$$\frac{\partial^2 c^{(a)}_{ijkl}}{\partial E_a^2} = \frac{2}{E_a^3}\left((1+\nu_a)\delta_{ik}\delta_{jl} - \nu_a(1+\kappa\nu_a)\delta_{ij}\delta_{kl}\right). \tag{77}$$

Then, the canonical system of the stress-based SFEM equations for both of these parameters randomized can be rewritten as

$$K^0_{\alpha\beta}a^0_\beta = F^0_\alpha, \tag{78}$$

$$K^0_{\alpha\beta}a^{,r}_\beta = -K^{,r}_{\alpha\beta}a^0_\beta, \tag{79}$$

$$K^0_{\alpha\beta}a^{(2)}_\beta = -\left(K^{,rs}_{\alpha\beta}a^0_\beta + 2K^{,r}_{\alpha\beta}a^{,s}_\beta\right)\mathrm{cov}(b_r, b_s), \tag{80}$$

since the fact that the external load vector is not a function of the input random variables vector introduced. Starting from the equations posed above, the computational implementation of the stress-based stochastic FEM based on the Hsieh–Clough–Tocher triangular or the Bogner–Fox–Schmit rectangular finite elements can be done. Further, considerations on the extension of the method presented on stochastic nonlinear statics or dynamics may be carried out starting from corresponding models for displacement-based SFEM models (Watwood and Hartz, 1968; Wieckowski, 1999).

4. Numerical illustration

4.1. Cantilever beam example by the second-order Airy functions

The general capabilities of the probabilistic second-order analysis are illustrated on the example of the homogeneous steel cantilever beam with unit thickness loaded by the shear force P (cf. Fig. 1). The example illustrates very well the usage of the second-order stochastic Airy functions to the stress analysis of plane elastostatics problems where the closed solution is available. Starting from the classical Airy functions theory (Timoshenko and Goodier, 1951), the stress and strain tensor components can be obtained as

$$\sigma_x = -\frac{3}{2}\frac{P}{c^3}xy, \quad \sigma_y = 0, \quad \tau_{xy} = -\frac{3P}{4c}\left(1-\frac{y^2}{c^2}\right), \tag{81}$$

$$\varepsilon_x = -\frac{3P}{2Ec^3}xy, \quad \varepsilon_y = \frac{3\nu P}{2Ec^3}xy, \quad \gamma_{xy} = -\frac{3(1+\nu)P}{2Ec^3}\left(c^2-y^2\right). \tag{82}$$

Using approximations (33)–(35) for the stress tensor components as well as Eqs. (37) and (38) for the strain tensor, the expected values and cross-covariances for these state functions can be calculated explicitly as it is shown in the Appendix A; the analysis can be extended on cross-correlations and higher order probabilistic moments as well. Let us consider the structure with the following parameters: $c = 0.10$ m, $L = 0.50$ m,

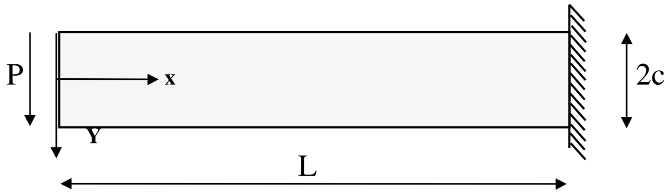

Fig. 1. Steel cantilever beam tested.

$P = 10$ kN, $E = 209$ GPa and $\nu = 0.3$. All these parameters except the length, L, are randomized separately to verify the influence of each parameter; standard deviations for all of these variables are taken as 10% of the corresponding expected values. Since the fact that expected values of stresses are quite closed to the relevant deterministic functions, variances are presented below by only. Figs. 2 and 3 contain second-order probabilistic moments of the variable σ_x obtained, thanks to second-order perturbation method implemented in MAPLE package Char et al., (1992) to compare variability of this component with respect to different input random parameters: external force and the height of the beam (horizontal axes of these graphs correspond to spatial coordinates x and y); all the computations are done by the use of symbolic differentiation tool built up in the program.

How it can be expected, the variance computed is equal to 0 for Young's modulus and Poisson's ratio since the fact that first part of Eq. (81) does not contain these variables. In the case of external force and beam height randomized (cf. Figs. 2 and 3) resulting extremal coefficients of variation are approximately equal to input variation of input random parameters. The decisive points of the structure from the reliability analysis point of view are the upper and lower edge of the clamped cross-section of a beam – standard deviation reaches its maximal value in this region.

Moreover, the comparison of the stochastic second-order method computations with the results of MCS technique are presented in Figs. 4–11. The expected values and standard deviations of strain tensor components ε_x and ε_y are pairly compared for SFEM and MCS computations in the function of spatial coordinates defining the beam plane. Young's modulus of the structure is taken as an input random variable while the total number of random trials in the MCS is assumed as equal to 10^3. For the clarity of the graphs, the coordinate x is scaled by 10^1 in Figs. 4–7 and by 10^2, in Figs. 8–11, the expected values and standard deviations collected are scaled by the same multiplicator. All the results illustrate very good agreement between the perturbation and simulation based analyses – the shapes of the corresponding functions are exactly the same while particular values are quite similar (with respect to the precision

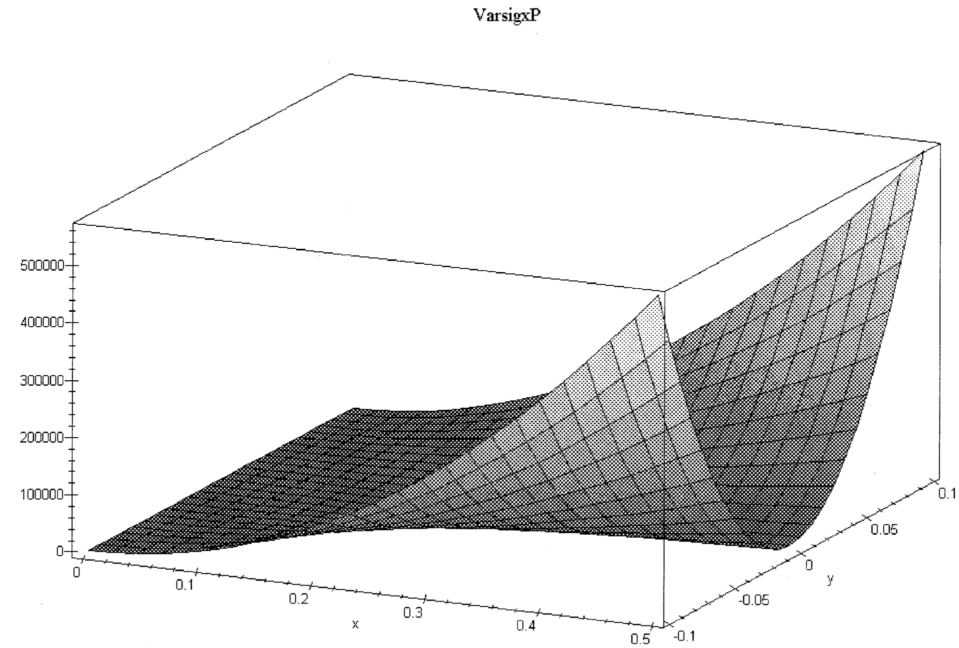

Fig. 2. Variance of σ_x for $P = P(\omega)$.

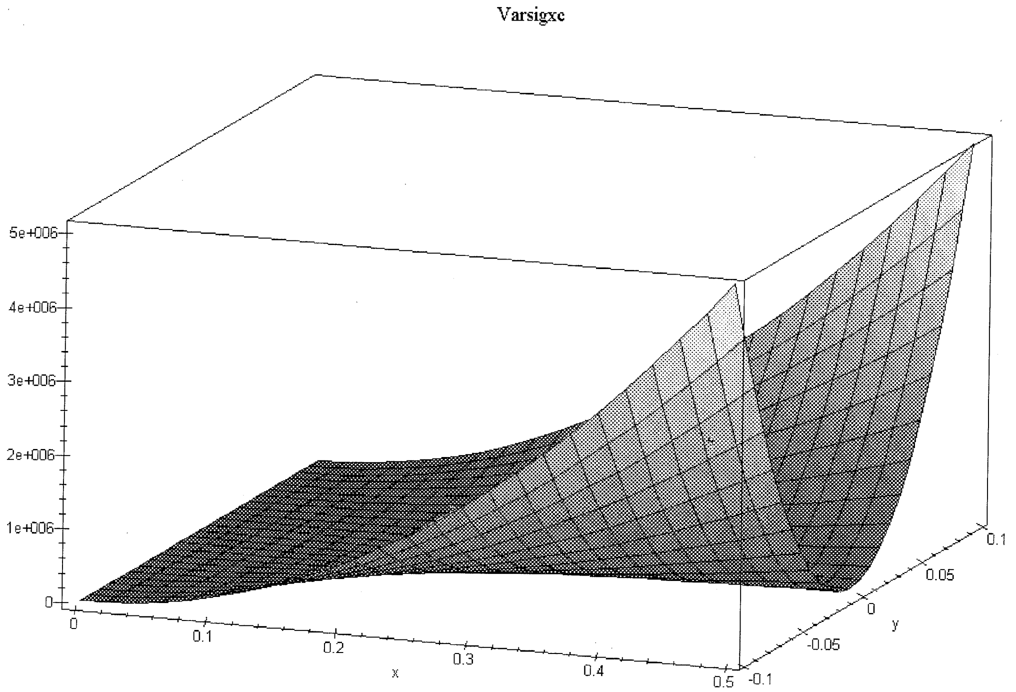

Fig. 3. Variance of σ_x for $c = c(\omega)$.

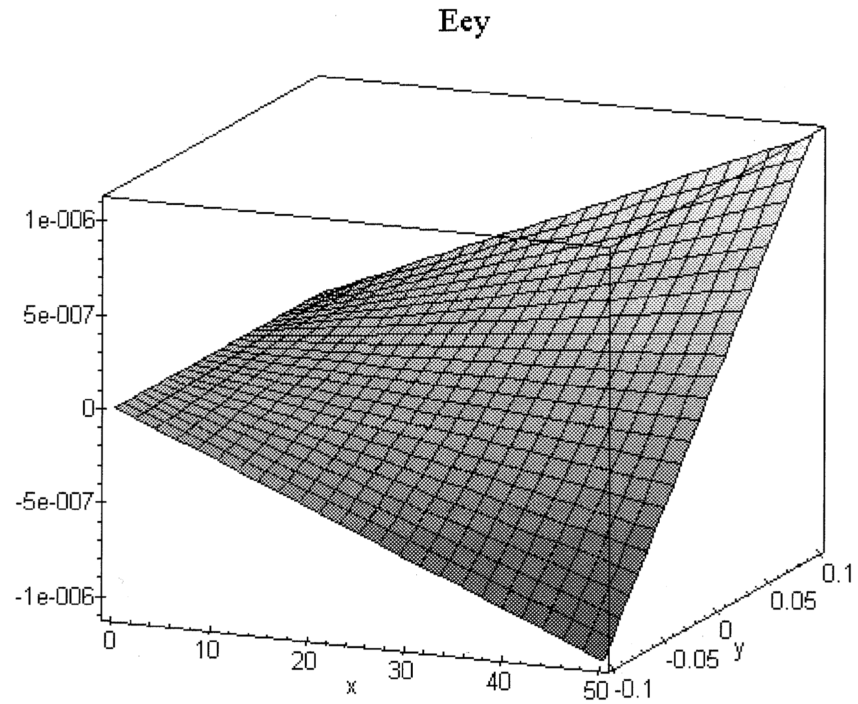

Fig. 4. Expected value of ε_y for $e = e(\omega)$.

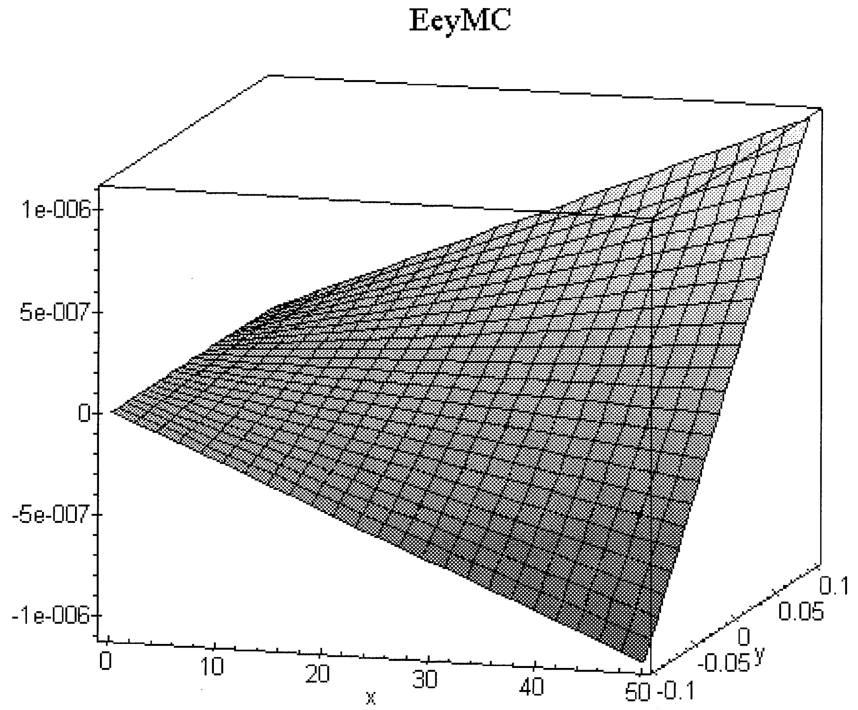

Fig. 5. Expected value of ε_y for $e = e(\omega)$, MCS.

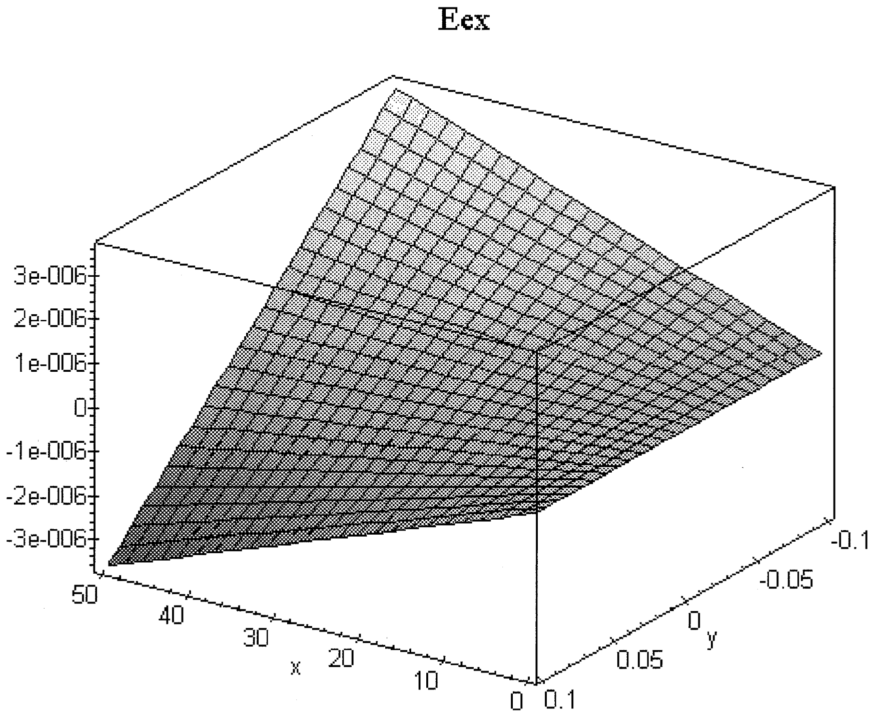

Fig. 6. Expected value of ε_y for $e = e(\omega)$.

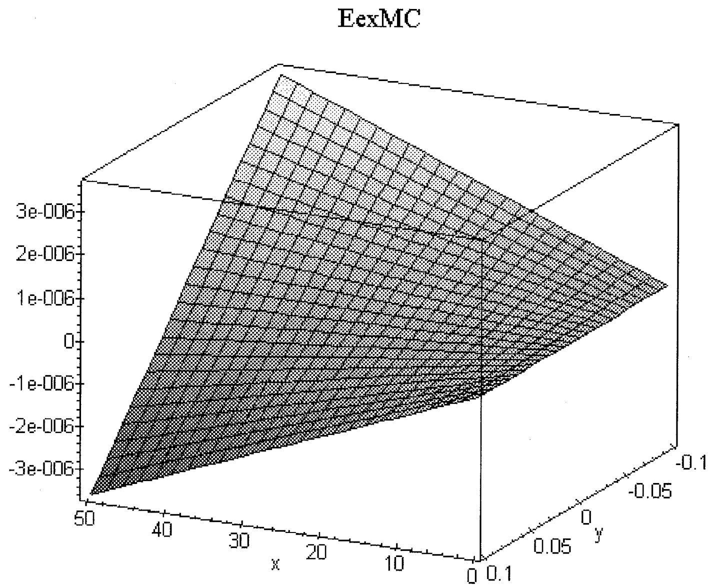

Fig. 7. Expected value of ε_x for $e = e(\omega)$, MCS.

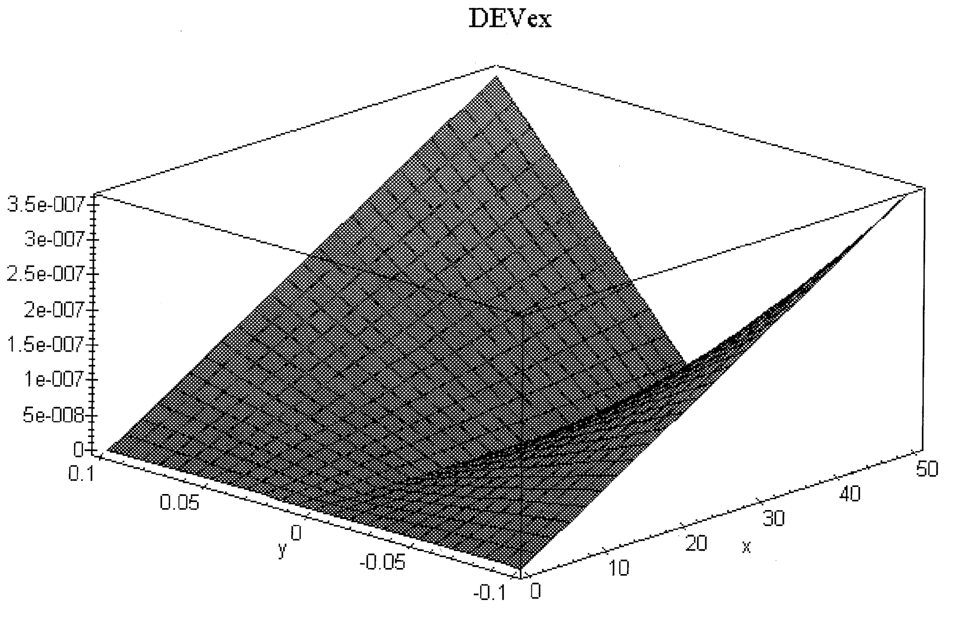

Fig. 8. Standard deviation of ε_x for $e = e(\omega)$.

presented on all vertical axes). Further, analogously to the previous results, higher order probabilistic moments are equal to 0 for the free edge and horizontal axis, while both moments analyzed have their maxima for upper and lower edge of a tree cross-section.

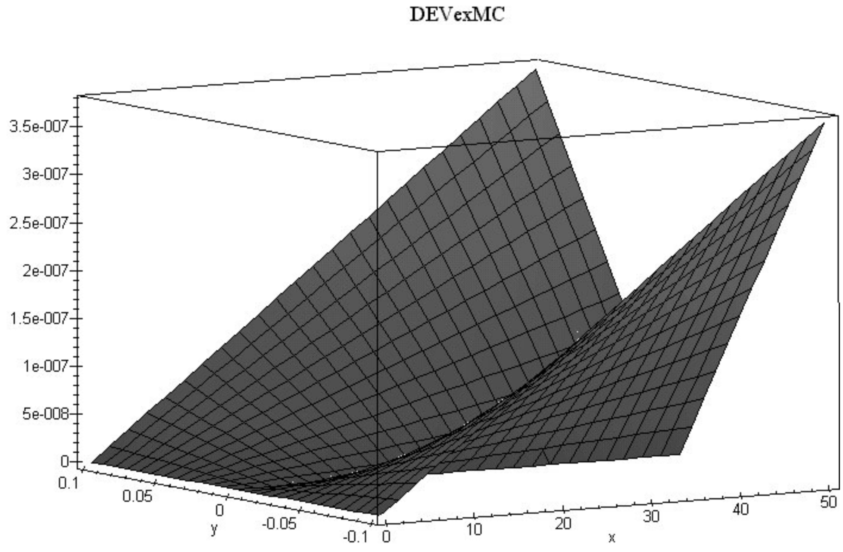

Fig. 9. Standard deviation of ε_x for $e = e(\omega)$, MCS.

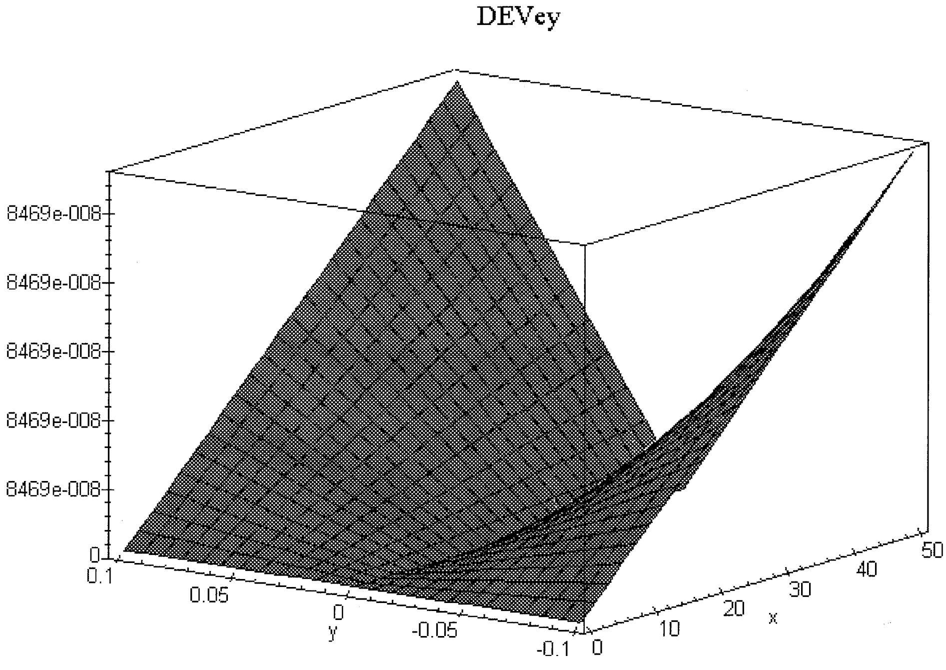

Fig. 10. Standard deviation of ε_y for $e = e(\omega)$.

It should be mentioned that the main value of the method proposed is capable of numerical modeling of randomness in structural geometry, which is much more complicated in the SFEM displacement-based analysis (Kaminski and Hien, 1999; Kaminski and Kleiber, 2000; Kleiber and Hien, 1992; Liu et al., 1986).

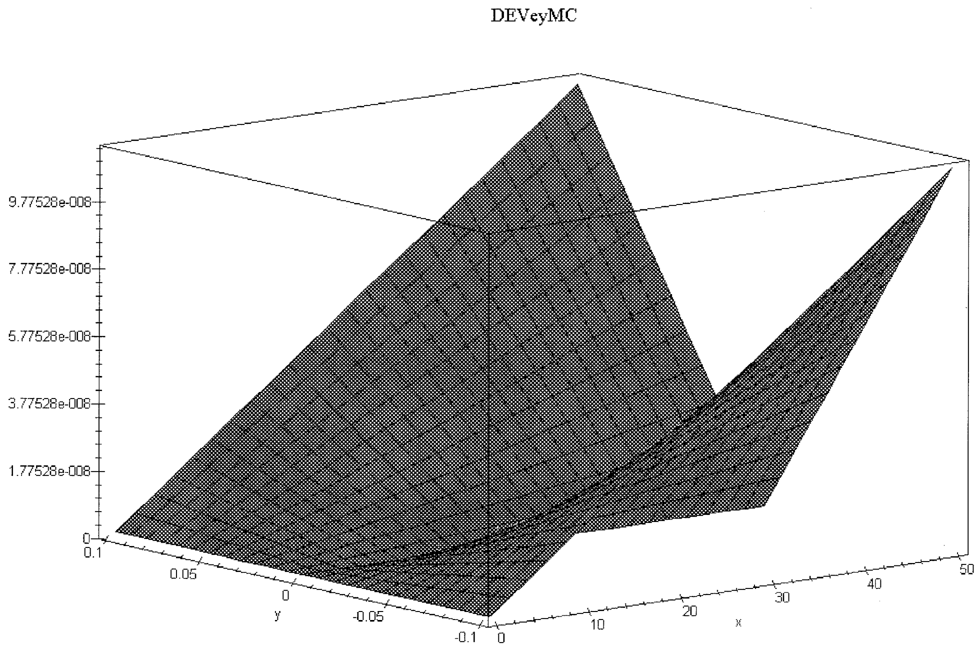

Fig. 11. Standard deviation of ε_y for $e = e(\omega)$, MCS.

Considering the fact that the results of a perturbation approach depend strongly on the coefficient of variation of input random variables, the additional computational studies should be carried out to compare the values resulting from MCS and SFEM analyses in terms of increasing variation coefficients that is outlined in the next computational experiment.

4.2. Torsion of a rectangular beam

The stochastic FEM implementation aspects are explained in details on the example of torsion of a square beam shown below. The discretization by the use of CST finite elements of a cross-section quarter is presented in Fig. 12. The Kirchhoff modulus is considered here as input random variable and is defined by its expected value $E[G]$ and variance var(G), while twisting angle is taken as $\theta = 1$. Solution of the SFEM equations (78)–(80) is obtained in the form of zeroth-, first- and second-order stress function components as follows:

$$\varphi_5^0 = 1,33334G^{-1}, \qquad \varphi_1^0 = 2,66667G^{-1},$$

$$\varphi_5^{;G} = -1,33334G^{-2}, \qquad \varphi_1^{;G} = -2,66667G^{-2},$$

$$\varphi_5^{;GG} = 2,66667G^{-3}, \qquad \varphi_1^{;GG} = 5,33332G^{-3}.$$

Therefore, the expected values and variances are equal to

$$E[\varphi_5] = 1,33334\left(E^{-1}[G] + E^{-3}[G]\mathrm{var}(G)\right), \quad \mathrm{var}(\varphi_5) = 1,77778\mathrm{var}(G)E^{-4}[G],$$

$$E[\varphi_1] = 2,66667\left(E^{-1}[G] + E^{-3}[G]\mathrm{var}(G)\right), \quad \mathrm{var}(\varphi_1) = 7,1111\mathrm{var}(G)E^{-4}[G].$$

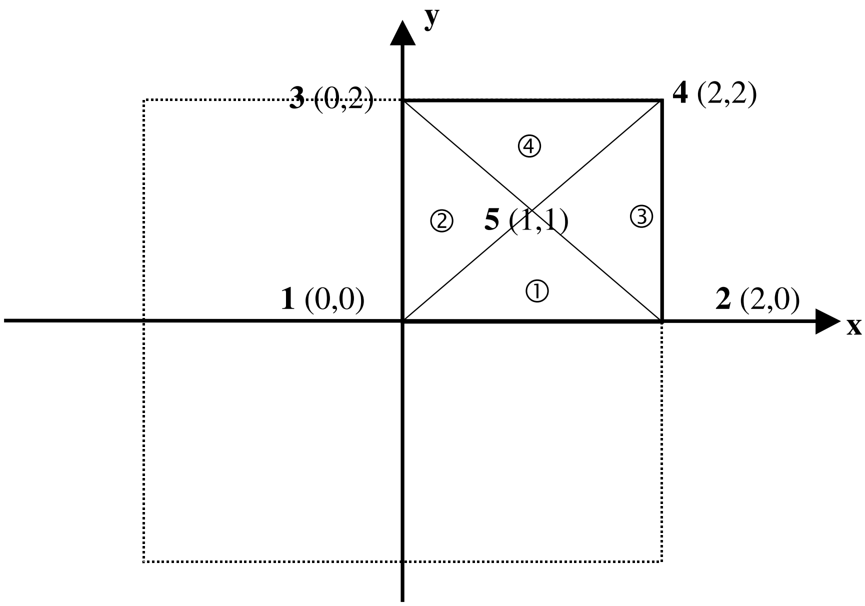

Fig. 12. Cross-section of the bar under torsion.

The corresponding probabilistic moments of the stresses in finite elements are equal to

- element no 1

$$E\left[\left\{\begin{matrix}\tau_{xz}\\ \tau_{yz}\end{matrix}\right\}\right] = E\left[\left\{\begin{matrix}\varphi_{,y}\\ -\varphi_{,x}\end{matrix}\right\}\right] = \left[\left\{\begin{matrix}\varphi^{0}_{,y} + \frac{1}{2}\varphi^{,GG}_{,y}\mathrm{var}(G)\\ -\varphi^{0}_{,x} - \frac{1}{2}\varphi^{,GG}_{,x}\mathrm{var}(G)\end{matrix}\right\}\right] = \left\{\begin{matrix}0\\ 1,33334\left[E^{-1}[G] + \frac{1}{2}E^{-3}[G]\mathrm{var}(G)\right]\end{matrix}\right\} = \left\{\begin{matrix}0\\ A\end{matrix}\right\},$$

$$\mathrm{var}\left[\left\{\begin{matrix}\tau_{xz}\\ \tau_{yz}\end{matrix}\right\}\right] = \mathrm{var}\left[\left\{\begin{matrix}\varphi_{,y}\\ -\varphi_{,x}\end{matrix}\right\}\right] = \left[\left\{\begin{matrix}\varphi^{,G}_{,y}\varphi^{,G}_{,y}\mathrm{var}(G)\\ \varphi^{,G}_{,x}\varphi^{,G}_{,x}\mathrm{var}(G)\end{matrix}\right\}\right] = \left\{\begin{matrix}0\\ 1,77778[E^{-4}[G]\mathrm{var}(G)]\end{matrix}\right\} = \left\{\begin{matrix}0\\ B\end{matrix}\right\},$$

- element no 2

$$E\left[\left\{\begin{matrix}\tau_{xz}\\ \tau_{yz}\end{matrix}\right\}\right] = \left\{\begin{matrix}-A\\ 0\end{matrix}\right\}, \qquad \mathrm{var}\left[\left\{\begin{matrix}\tau_{xz}\\ \tau_{yz}\end{matrix}\right\}\right] = \left\{\begin{matrix}B\\ 0\end{matrix}\right\},$$

- element no 3

$$E\left[\left\{\begin{matrix}\tau_{xz}\\ \tau_{yz}\end{matrix}\right\}\right] = \left\{\begin{matrix}0\\ A\end{matrix}\right\}, \qquad \mathrm{var}\left[\left\{\begin{matrix}\tau_{xz}\\ \tau_{yz}\end{matrix}\right\}\right] = \left\{\begin{matrix}0\\ B\end{matrix}\right\},$$

- element no 4

$$E\left[\left\{\begin{matrix}\tau_{xz}\\ \tau_{yz}\end{matrix}\right\}\right] = \left\{\begin{matrix}-A\\ 0\end{matrix}\right\}, \qquad \mathrm{var}\left[\left\{\begin{matrix}\tau_{xz}\\ \tau_{yz}\end{matrix}\right\}\right] = \left\{\begin{matrix}B\\ 0\end{matrix}\right\}.$$

Finally, the cross-sectional twisting moment M and its probabilistic moments are obtained as

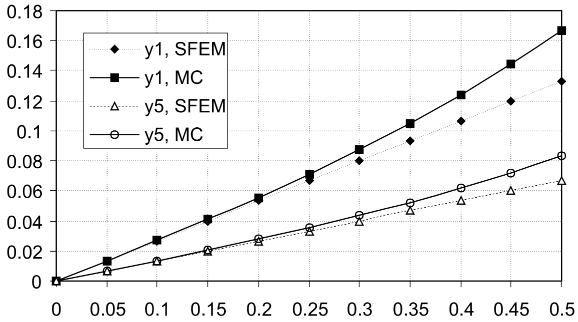

Fig. 13. Convergence studies on the solution coefficients of variation.

$$M = \int \int_{\Omega} \varphi \, dx \, dy, E[M] = 7,1111\left(E^{-1}[G] + 0.5E^{-3}[G]\mathrm{var}(G)\right), \mathrm{var}(M) = 50,5677E^{-4}[G]\mathrm{var}(G).$$

To illustrate the accuracy of the second-order approach, the variances of φ_1 and φ_5 (denoted by y_1 and y_5) are compared with corresponding values computed by the use of MCS technique and corresponding maximum-likelihood statistical estimators (Bendat and Piersol, 1971). The results for $E[G] = 10.0$ and 10^3 being the total number of the MCS random samples are collected in Fig. 13 (denoted by y_1 and y_5) as a function of var(G) in the range corresponding to the coefficient of variation from the interval (0.0; 0.5).

Due to the results of previous computational experiments (Kaminski and Hien, 1999; Kaminski and Kleiber, 2000; Kleiber and Hien, 1992; Liu et al., 1986), the coefficients of variation obtained for the MCS are generally greater than those calculated in the stochastic second-order perturbation analysis. These differences are negligible for $\alpha(G) \leqslant 0.2$; however, for coefficients tending to 0.5 are even in the range of 25% (for $\varepsilon = 1$). It is considered that the method proposed may be used in further computations with the restriction on the coefficient of input random variables.

5. Concluding remarks

(1) The results obtained show, in the context of stress tensor probabilistic moments (especially variances), that the stress-based stochastic FEM formulated above makes it possible to relatively simply calculate all of these moments that is complicated in the case of displacement based, traditional formulation of the SFEM. Further, comparing various computational tests for both of these methods, it is visible that using the displacement-based SFEM, it is possible to randomize material properties of the structure considered, while the geometrical randomness can be relatively easy implemented in stress-based SFEM version using additional Airy functions. In further computational studies, the stochastic structural sensitivity analysis should be carried out with respect to the expected values and variances of various structural parameters. It can be done by the use of the adjoint variable method or the direct differentiation method approximations (Kleiber et al., 1997) formulated in the context of the complementary energy principle (Azene, 1979; Tabarrok, 1984) in the conjunction with stochastic perturbation approach.

(2) Starting from the second-order probabilistic equations for the stress-based stochastic FEM and, using the classical Hsieh–Clough–Tocher triangular or the Bogner–Fox–Schmit rectangular finite elements (Watwood and Hartz, 1968; Wieckowski, 1999), the corresponding general computer program can be implemented. Since the fact that the SOSM extension of deterministic FEM does not need any intervention

within the finite elements subroutines, various existing stress-based computer routines (Wieckowski, 1999) can be used to the proposed stochastic second-order implementation. On the other hand, by the analogy to the considerations presented in this paper, the SOSM approach to the torsion or related field problems can be introduced by using the Prandtl function proposed above or so-called flux method (Fraejis de Veubeke and Hogge, 1972). Both SOSM formulations may be useful in stochastic second-order homogenization analysis of composite materials (Kaminski, 2000; Kaminski and Kleiber, 2000).

(3). Taking into account the analysis presented in this paper, it can be observed that displacement-based FEM can be recommended for computations of the displacement field probabilistic moments, while the expected values and cross-covariances of stresses are more efficiently computed using the approach proposed above, which can be useful in reliability analysis. Analogously to the displacement-based FEM, the maximum value of any input coefficient of variation should not be larger than 0.2 (Kleiber and Hien, 1992; Liu et al., 1986). On the other hand, significant time savings are obtained in comparison to the MCS and spatial discretization of random fields contrary to the stochastic spectral techniques (Ghanem and Spanos, 1991).

Acknowledgement

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Appendix A

The second-order perturbation method is used to derive the stress and strain tensor components in the bar. Starting from Eqs. (81 and 82) and relations for expected values and variances of strain and stress tensor components (cf. Eqs. (31)–(34) and (36)–(38)), we obtain for $c = c(w)$

- the expected values:

$$E[\sigma_x] = -\frac{3}{2}\frac{P}{E^3[c]}xy - \frac{9P}{E^5[c]}xy\,\mathrm{var}(c), \tag{A.1}$$

$$E[\tau_{xy}] = -\frac{3P}{4E[c]}\left(1 - \frac{y^2}{E^2[c]}\right) + \frac{1}{2}\left(-\frac{3P}{2E^3[c]} + \frac{9Py^2}{E^5[c]}\right)\mathrm{var}(c), \tag{A.2}$$

$$E[\varepsilon_x] = -\frac{3P}{2EE^3[c]}xy - \frac{9Pxy}{EE^5[c]}\mathrm{var}(c), \tag{A.3}$$

$$E[\varepsilon_y] = \frac{3vP}{2EE^3[c]}xy + \frac{9vPxy}{EE^5[c]}\mathrm{var}(c), \tag{A.4}$$

$$E[\gamma_{xy}] = -\frac{3(1+v)P}{2EE^3[c]}(c^2 - y^2) + \frac{1}{2}\left(\frac{-3(1+v)P}{EE^3[c]} + \frac{18(1+v)Py^2}{EE^5[c]}\right)\mathrm{var}(c), \tag{A.5}$$

- the variances:

$$\mathrm{var}(\sigma_x) = -\frac{81}{4}\frac{Px^2y^2}{E^8[c]}\mathrm{var}(c), \tag{A.6}$$

$$\mathrm{var}(\tau_{xy}) = \left(\frac{3P}{4E^2[c]} - \frac{9Py^2}{4E^4[c]}\right)^2 \mathrm{var}(c), \tag{A.7}$$

$$\mathrm{var}(\varepsilon_x) = \left(\frac{9Pxy}{2EE^4[c]}\right)^2 \mathrm{var}(c), \tag{A.8}$$

$$\mathrm{var}(\varepsilon_y) = \left(\frac{9\nu Pxy}{2EE^4[c]}\right)^2 \mathrm{var}(c), \tag{A.9}$$

$$\mathrm{var}(\gamma_{xy}) = \left(\frac{3(1+\nu)P}{2EE^2[c]} - \frac{9(1+\nu)Py^2}{2EE^4[c]}\right)^2 \mathrm{var}(c). \tag{A.10}$$

By the analogous way, the cross-covariances between any of these components can be derived as well as the expected values and variances of the same tensors for material properties of the structure and external load treated as the input random variables.

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